8 Conic Sections

SYLLABUS OUTLINE

Areas to be studied:

• Terminology for conics. • The ellipse, parabola and hyperbola as sections of a right cone.
• Understanding of focal points, focal sphere, directrix and eccentricity in the context of conic sections.
• *Derivation of focal points, directrix and eccentricity using the focal sphere and solid cone.*
• Construction of conic curves as geometric loci. • Geometric properties common to the conic curves.
• Tangents to conics.
• *Construction of hyperbolae from focal points and transverse axis.*

Learning outcomes

Students should be able to:

Higher and Ordinary levels

- Understand the terms used in the study of conics, e.g. chord, focal chord directrix, vertex, ordinate, tangent, normal, major and minor axes/auxiliary circles, eccentricity, transverse axis.
- Construct ellipse, parabola, hyperbola as true sections of solid cone.
- Construct the conic sections, the ellipse, parabola and hyperbola, as plane loci from given data relating to eccentricity, foci, vertices, directrices and given points on the curve.
- Construct ellipse, parabola and hyperbola in a rectangle given principal vertice(s).
- Construct tangents to the conic sections from points on the curve.

Higher level only

- *Understand the terms used in the study of conics, double ordinate, latus rectum, focal sphere etc.*
- *Construct ellipse, parabola, hyperbola as true sections of solid cone and derive directrices, foci, vertices and eccentricity of these curves.*
- *Construct tangents to the conic sections from points outside the curve.*
- *Construct a double hyperbola given the foci and a point on the curve, or given the length of the transverse axis and the foci.*
- *Determine the centre of curvature and evolute for conic sections.*

Parabola as a Section of a Cone

A parabola is produced by slicing a cone with a plane that is parallel to a side of the cone in elevation.

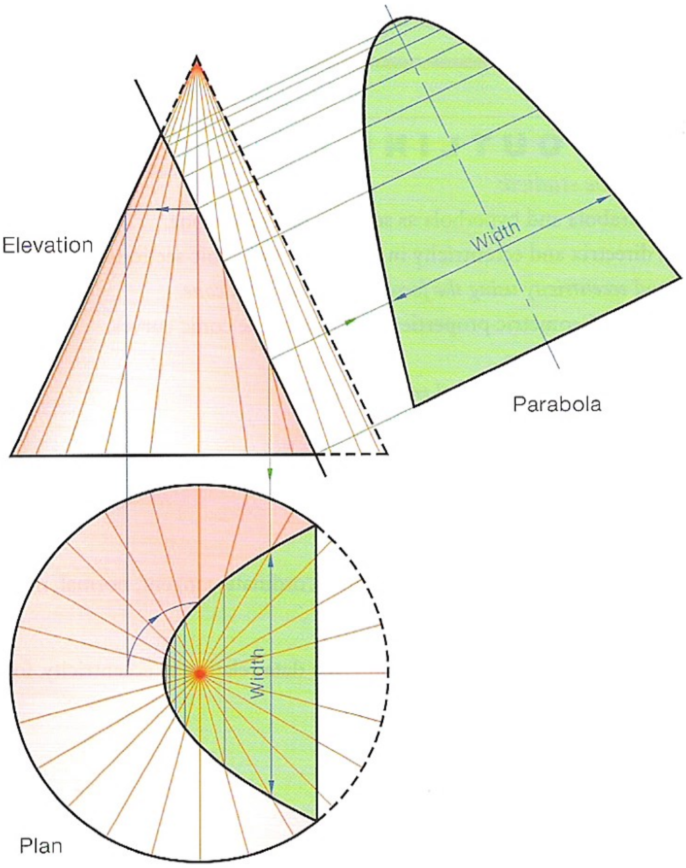

Fig. 8.1

CONSTRUCTION

(1) Draw the plan and elevation of a cone.
(2) Divide the plan into a number of generators which are projected onto the elevation.
(3) Draw in the cutting plane parallel to the side of the cone.
(4) As the generators are cut in elevation they are projected down to the corresponding elements in plan, producing a curve as shown in Fig. 8.1.
(5) The parabola is produced by finding the true shape of this section. Projection lines are produced perpendicular to the section plane and widths are taken from the plan.

Focal Sphere for a Parabola

HIGHER LEVEL

If a sphere is inserted into the tip of the cone so that it touches the side of the cone and also the cutting plane we have the **focal sphere**. This sphere touches the cutting plane at one point only, the focus, hence the name. The sphere also touches the cone all the way round to form a circle. If this circle is extended to form a plane, where the two planes intersect will give the directrix, Fig. 8.3.

The focal sphere can be constructed for the other two conics in a similar fashion as we will see later in this chapter.

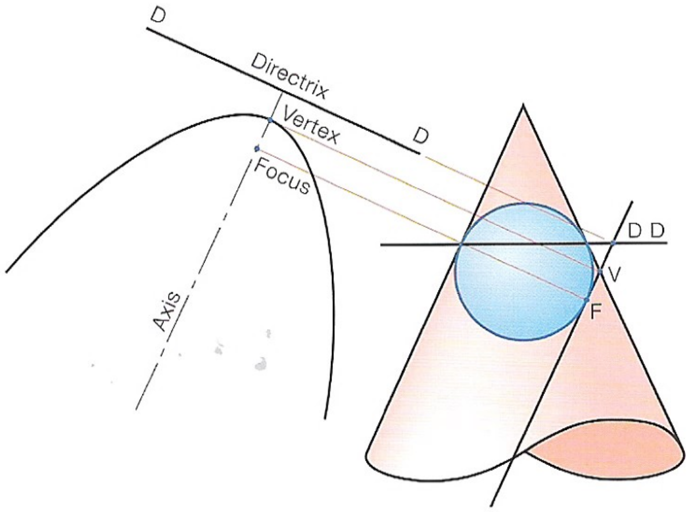

Fig. 8.2

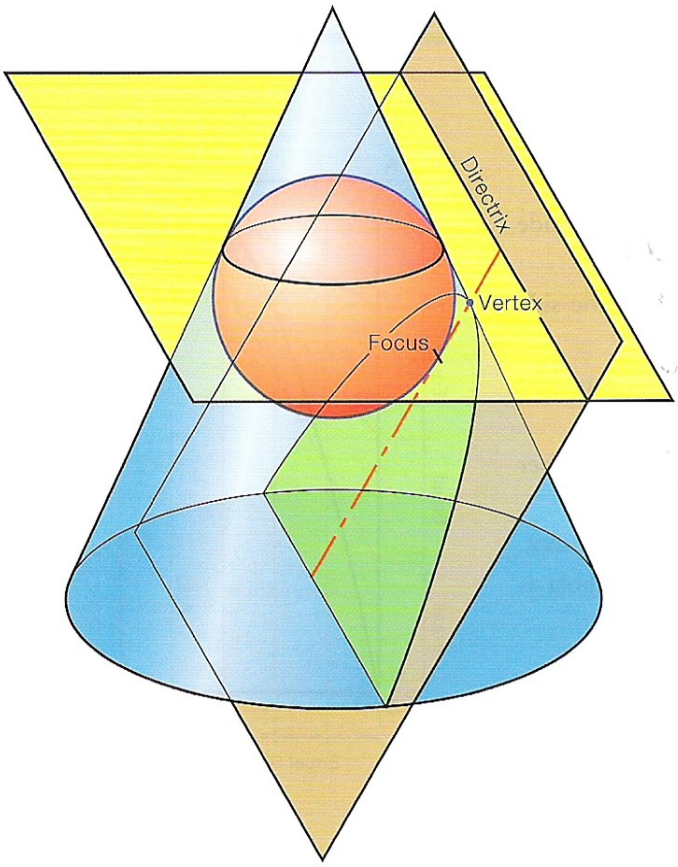

Fig. 8.3

Note: As well as being seen as a conic section, a parabola can also be seen as a locus or path of a point, such that, it is at all stages equidistant from a given line (directrix) and a given point (focus).

Terminology

Chord – A straight line touching the curve in two places.

Focal Chord – As above, but also passing through the focus.

Latus Rectum – Special focal chord perpendicular to the axis.

Ordinate – A line perpendicular to the axis, starting on the axis and ending on the curve.

Directrix – Line of intersection between the section plane and the plane formed by the intersecting points of the focal sphere and the cone.

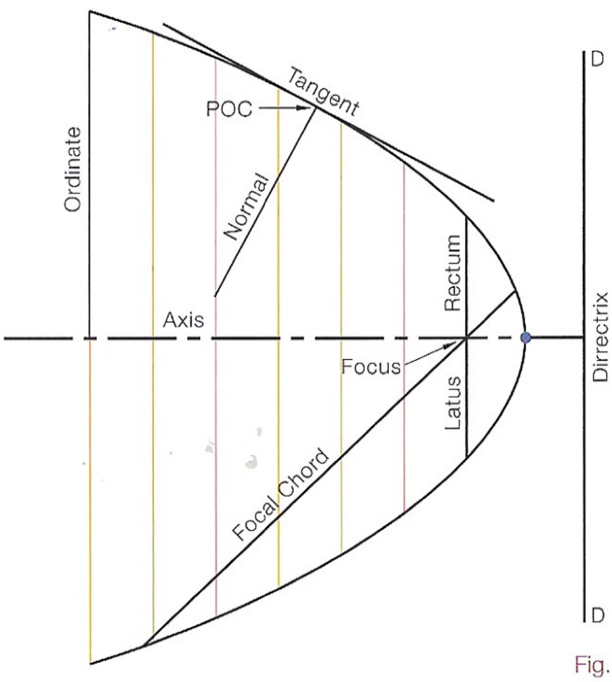

Fig. 8.4

Three Methods of Constructing a Parabola

Rectangle Method

(1) Divide one of the sides in half to find the vertex, e.g. side AD.

(2) Draw the axis through V and perpendicular to the side AD. The rectangle is now halved.

(3) Divide edge AB into any number of **equal** divisions and join each up to V as shown.

(4) Point A to the vertex is now divided into the same number of equal divisions.

(5) Lines are drawn from these points parallel to the axis. Where the two sets of lines intersect, plot the parabola as shown in Fig. 8.5.

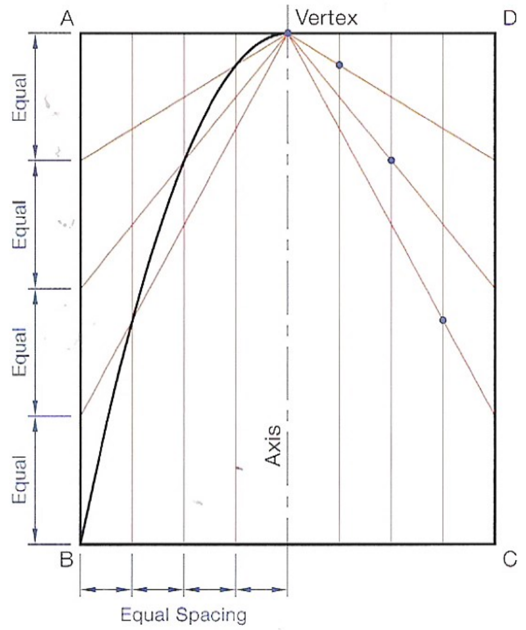

Fig. 8.5

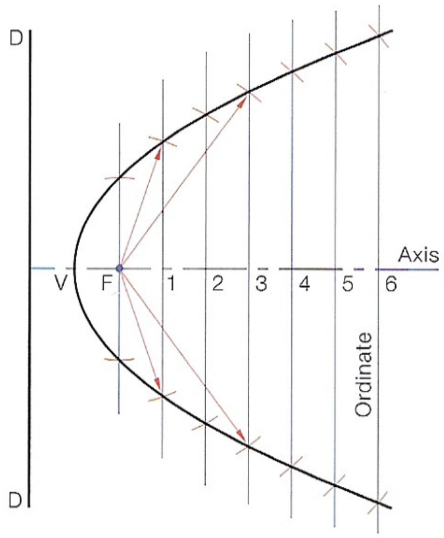

Fig. 8.6

Compass Method

(1) Find the vertex halfway between focus and directrix.

(2) Draw a series of ordinates, F to 6.

(3) Take a radius from the directrix DD to the first ordinate at F.

(4) Move the compass point to the **focus** and scribe arcs above and below the axis on this ordinate at F.

(5) Next take a radius from the directrix to ordinate 1.

(6) Move the compass point to the **focus** and scribe arcs to cut ordinate 1 above and below the axis.

(7) Continue as necessary remembering to always draw the arcs having F as centre.

Eccentricity Method

The eccentricity for a parabola is always **equal to 1** or **1/1.**
Eccentricity is a ratio between the two distances:
(i) from the focus to a point P on the curve,
(ii) from the same point on the curve to the directrix.

$$\text{Eccentricity} = \frac{\text{Distance from focus to a point}}{\text{Point to directrix}} = \frac{\text{F to P}}{\text{P to DD}}$$

Since the eccentricity of a parabola is unity, any point on the parabola must be equidistant from the directrix and focus.

The eccentricity line for a parabola will always be at 45° to the axis.

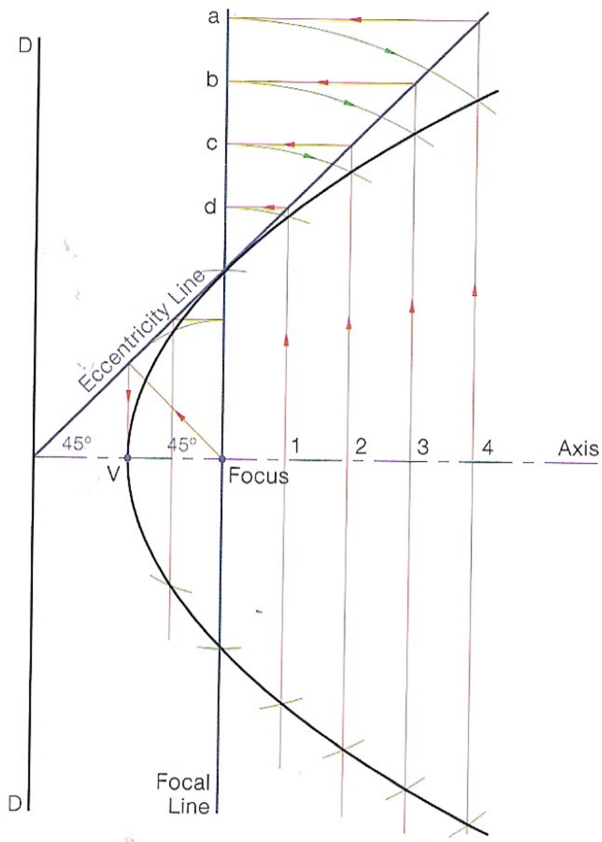

Fig. 8.7

CONSTRUCTION

Given directrix, axis and focus.

(1) Set up the eccentricity line. This will be a 45° line for a parabola.
(2) The vertex is found by projecting up from the focus at 45° to the axis to hit the eccentricity line and down perpendicular to the axis, giving V.
(3) Draw a perpendicular to the axis through the focus giving the focal line.
(4) Where the eccentricity line and focal line intersect is a point on the curve which can be swung below the axis.
(5) Draw lines **1, 2, 3, 4...** up to the eccentricity line, perpendicular to the axis, and then project across to the focal line, parallel to the axis. This finds points **a, b, c, d...**
(6) With the focus as centre, scribe an arc from a to hit line **1** above and below the axis.
(7) With the focus as centre, scribe an arc from **b** to hit line **2** above and below the axis.

Continue in this manner to plot more points on the curve.

Tangents to a Parabola from a Point on the Curve

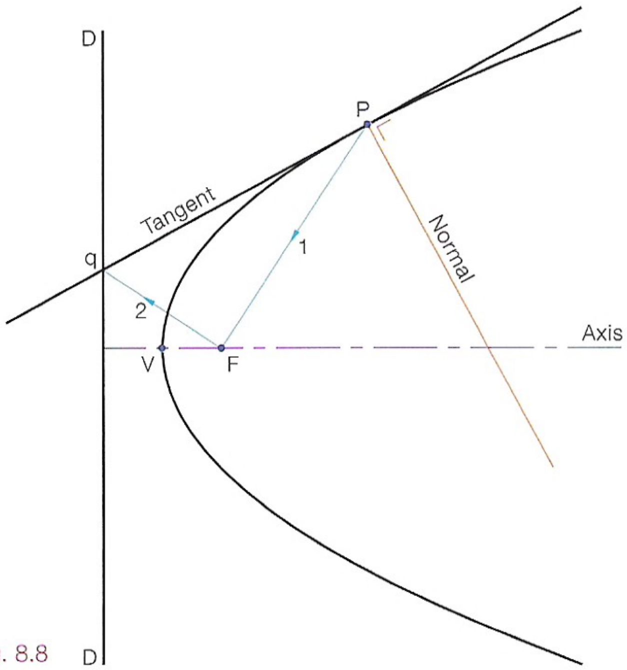

Fig. 8.8

Method 1

Fig. 8.8

(1) Join point P on the curve to the focus F.

(2) At F create a 90° angle and extend to hit the directrix DD at q.

(3) Point q is a point on the tangent.

(4) Join P to q to give the tangent.

A perpendicular to the tangent at P, the point of contact, will give the normal at P.

Method 2

Fig. 8.9

(1) Draw a perpendicular line to the axis from point P to give point r.

(2) With the vertex as centre rotate point r to give q.

(3) Join P to q to form the tangent.

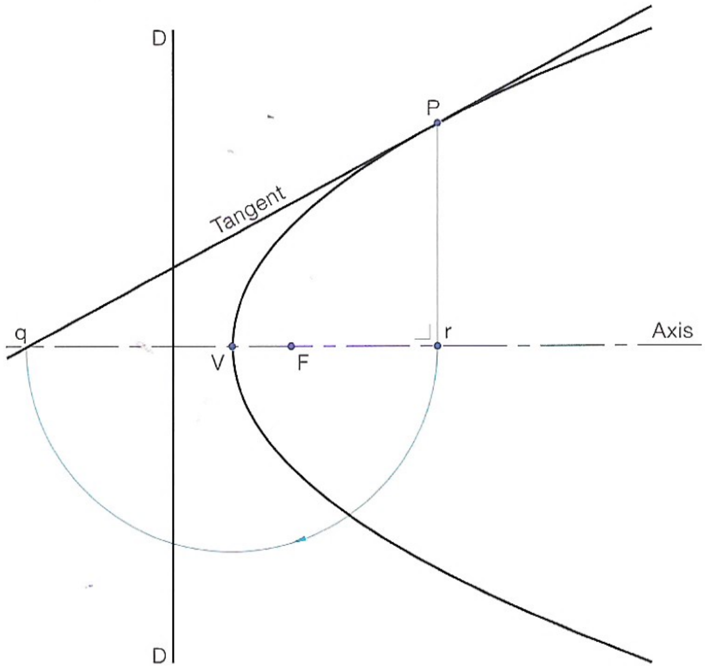

Fig. 8.9

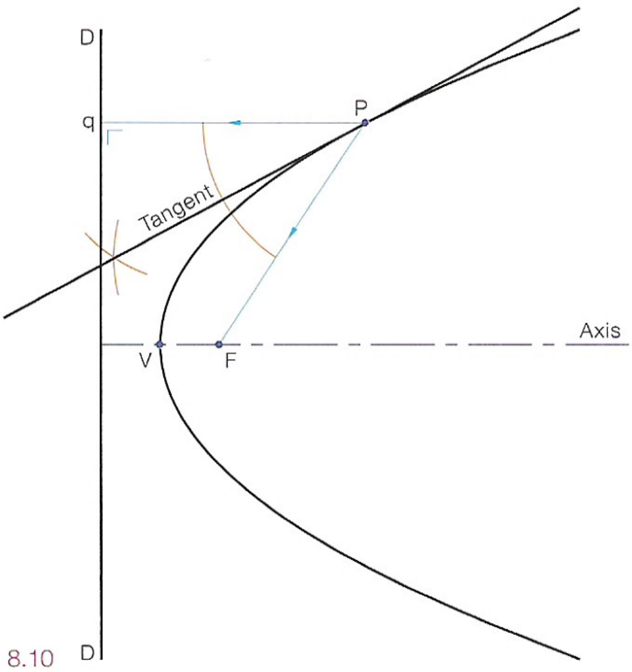

Fig. 8.10

Method 3

Fig. 8.10

(1) Join P on the curve to the focus F.

(2) Draw a line from P parallel to the axis to hit the directrix at q.

Note: This line Pq can be considered to be a line joining to a focal point at infinity and hence will tie in with one of the methods of constructing tangents for the hyperbola and ellipse.

(3) The bisector of the angle formed, qPF will give the tangent.

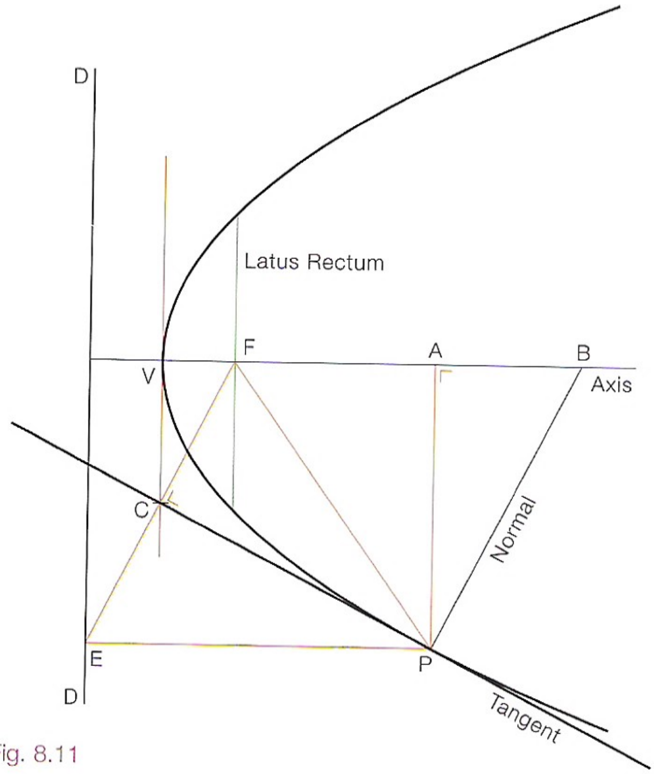

Fig. 8.11

Points of interest about the parabola. Fig. 8.11

The latus rectum equals 4FV.
Length from A to B equals half the latus rectum.
Length CV equals half of PA.
Length FB equals EP.
Length FC equals CE.

Given a parabola to locate the focus and the directrix. Fig. 8.12

(1) Draw any ordinate and extend so that AB equals twice AV.
(2) Join B to the vertex V locating point C on the curve.
(3) C will always be a point on the end of the latus rectum. Drop C perpendicular to the axis to locate F the focus.
(4) For a parabola the eccentricity is always one, so VF equals V,DD.
Draw the directrix.

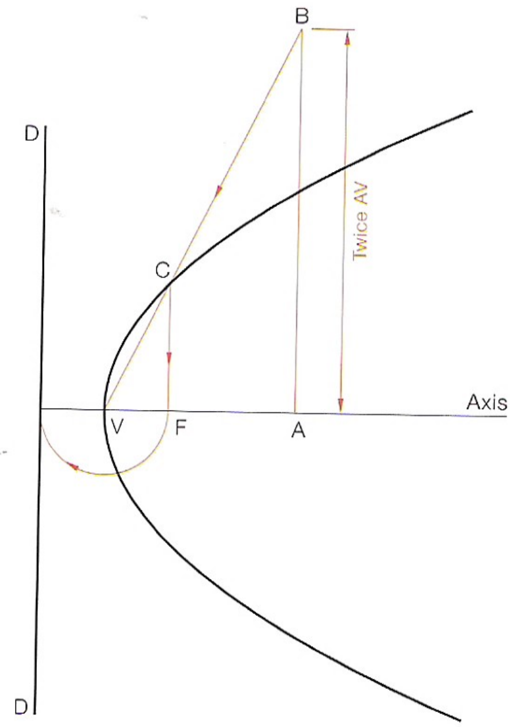

Fig. 8.12